



Maximizing a probability: A student workshop on an application of continuous distributions

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Abstract

For many students meeting, say, the gamma distribution for the first time, it may well turn out to be a rather fruitless encounter unless they are immediately able to see an application of this probability model to some real-life situation. With this in mind, we pose here an appealing problem that can be used as the basis for a workshop activity introducing, and subsequently encouraging the exploration of, many of the well-known continuous distributions in a meaningful way. We provide suggestions as to how the session might be run, discuss any pedagogical issues that arise and highlight particularly interesting features of the distributions.

1. Introduction

Consider the following scenario: A group of friends are away camping for the weekend and decide to go for a walk after dark. It is so dark in fact that each of them brings a torch. They all turn their torches on at the start of the walk, head out to some point and then retrace their steps to get back to the tents. Some torches die on the way out, others die on the way back, while some are still shining at the end of the walk. A non-trivial and intriguing question is: How long should their walk be in order that the expected number of torches dying out on the return journey is maximized?

In this article we provide a detailed account of a workshop activity, the aim of which was to investigate the above problem. However, before describing both the resulting mathematics and the pedagogical issues that arose, I should explain that the idea for this activity did actually come about somewhat by accident. During a recent lesson with one of my classes in which the students were, via some rather routine examples from a textbook, familiarizing themselves with the tabulated cumulative distribution function of the *standard normal* random variable $Z \sim N(0,1)$, I asked them to find $q > 0$ such that

$P(q < Z < 2q) = 0.15$. Immediately after having posed this question, it occurred to me that there might not actually be a solution. This led me to consider the problem of finding the largest number p such that $P(q < Z < 2q) = p$ for some value of $q > 0$.

Using the result

$$\frac{d}{dx} \int_{u(x)}^{v(x)} f(t) dt = f\{v(x)\} \frac{dv}{dx} - f\{u(x)\} \frac{du}{dx} \quad (1)$$

from Spiegel (1963, p. 163), and noting that the pdf for the standard normal distribution is given by

$$f(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}x^2\right), \quad -\infty < x < \infty,$$

we obtain

$$\frac{d}{dx} \int_x^{2x} \frac{\exp(-t^2/2)}{\sqrt{2\pi}} dt = \frac{1}{\sqrt{2\pi}} \{2\exp(-2x^2) - \exp(-x^2/2)\}.$$

From this it follows, on equating the expression on the right to zero, that $P(q < Z < 2q)$ possesses a stationary point when

$$q = \sqrt{\frac{2\log 2}{3}}.$$

To show that this probability is indeed maximized for the above value of q , we obtain the second derivative and evaluate it at this point to give

$$-\frac{1}{\sqrt[3]{2}} \sqrt{\frac{3\log 2}{\pi}}.$$

In this case we find that $P(q < Z < 2q) \approx 0.1613$, so the task I set my class of finding $q > 0$ such that $P(q < Z < 2q) = 0.15$ did indeed have a solution. The problem concerning the torches evolved from this relatively simple question.

In order to help set the scene mathematically, we offer here several additional comments regarding our probabilistic scenario, and define some notation that will be adopted in later sections. Let X be the random variable representing, in some appropriate unit of time, the ‘longevity’ of a torch. (It is clear that a number of modeling assumptions need to be made regarding X , but consideration of these is postponed until the following section.) From the preceding calculation we know that if X followed the standard normal distribution, then, in order to maximize the expected number of torches dying out on the return journey, the total length of the walk (in hours, say) would need to be $2q \approx 1.3596$. The standard normal distribution is of course totally unsuitable for modeling the longevity of a torch. In the workshop we considered a number of other well-known continuous distributions (as given in [Grimmett and Stirzaker \(2001, pp. 95-97\)](#), for example), each of which may or may not be deemed a potential candidate for X . Although our problem concerns finding $q > 0$ such that $P(q < X < 2q)$ is maximized, we do, for the sake of generality, maximize $P(q < X < kq)$ subject to $q > 0$ and $k > 1$. Where possible, exact expressions for these maxima are obtained in terms of k and the distribution parameters. The limiting cases that arise as $k \rightarrow 1$ and $k \rightarrow \infty$ are also considered.

2. The pedagogical aspect

I ran this session for a class of advanced undergraduates, each of whom attended on a voluntary basis. They were put into small groups in order to create opportunities for discussion, peer teaching and collaboration throughout the session. This particular set of students had already covered a first course in statistics and a more advanced calculus course. They were aware of the common continuous distributions, having looked at them briefly in the statistics course. However, they possessed neither real insight into specific mathematical properties of these distributions nor any experience with regard to assessing their suitability for a particular modeling situation.

In order to facilitate high quality learning, it seemed, bearing in mind the student-centered and relatively relaxed nature of the workshop setting, that the following points advocated by [Angelo and Cross \(1993\)](#) were particularly important:

- To help make their lessons successful, teachers must state their aims and objectives explicitly, and then obtain clear and specific feedback in order to ascertain the extent to which these aims and objectives are being achieved.
- For the students' learning, it is essential that they receive appropriate and focused feedback from the teacher.
- Teachers reflect on the learning issues encountered in the classroom and then consider how improvements may be made in order to enhance the students' learning experiences in future lessons.

To provide some initial impetus and motivation, the problem was introduced by way of a practical activity. We recreated scaled-down journeys by walking across the classroom and back. In order to simulate the lifetime of each torch, I used the statistical software [R \(2004\)](#) to sample from a [Weibull](#) distribution; more on specific distributions in due course. After having generated the torch lifetimes for each of the students, they were set off on their journeys one at a time. A clock indicated when the torch of a particular student had died out, at which point I asked them to stop where they were and to remain facing in the direction they had been walking. In this manner we were able to see exactly where each of the torches had failed on the journey.

In the initial simulation, I set the parameters so that the majority of torches died on the return journey, with a few failing on the way out and a few still shining at the end. Further simulations were carried out, from which it was possible to see how the distribution of the points at which the torches died changed dramatically as the parameters of the distribution were varied. Indeed, I have found that the visual impact of such a demonstration does allow even the weaker students fully to understand the nature of the problem under consideration. After this introduction, the aims of the workshop were made explicit to the students. These were:

- To nominate, with justification, a suitable probability model for the longevity of the torch.
- To select appropriate parameter values and hence calculate the length of the walk such that the expected number of torches dying out on the return journey is maximized?

There are of course a number of assumptions that need to be made when modeling this situation, and I next elicited, via verbal prompts, a number of these from the students. It was agreed, for the sake of simplicity, that the lifetime of each torch should follow the same continuous distribution, whatever that may be. We might thus suppose that each torch is of the same type, having identical and previously-unused bulbs and batteries. The students next decided that it would be sensible to assume that the outward journey should take exactly the same time as the return journey. They also acknowledged that one slight difficulty was the fact that torches tend to dim gradually as batteries lose their strength, and that

it might be rather difficult to say exactly when a torch is considered ‘dead’. We chose, however, to assume that it is indeed possible to state such times precisely. Finally, in order to proceed with any calculations, a suitable probability distribution is required to model the longevity of a torch. This is something we consider in detail in Section 3.

At this early phase of the workshop it was necessary to map out and construct some key learning paths, given the students’ base knowledge stated at the beginning of this section. Indeed, the following set of workshop objectives give an idea of how these learning paths may be created. The students needed first to reacquaint themselves with the notion of continuous random variables, and to recall how, via their pdfs, they may be used to model quantities such as the lifetime of a randomly-chosen torch. The next step was to gain confidence in using pdfs to calculate probabilities associated with random variables, and to be able to interpret the mean and variance of a particular distribution. They needed also to consider how the shapes of the various pdfs are transformed as the parameters are varied. Some independent research could then take place into common applications of each of the distributions under consideration, creating the possibility for informed decisions to be made about the suitability of each of these distributions with regard to modeling our particular probabilistic scenario. Finally, the students would also be required to employ the use of more advanced calculus in order to obtain and interpret results associated with our maximization problem. To aid some of the above, I ensured that computers with mathematical software and internet access were made available to the students.

Although I did not wish unduly to influence the progression of the workshop, I felt it important that I provided early and detailed feedback in order to avoid the students wasting too much time pursuing fruitless lines of enquiry. If there was a common misconception or error I would stop the workshop and facilitate a whole-class discussion. Otherwise, I would speak to individuals or groups as I walked round, assessing students either via straightforward knowledge-based questions such as “To which family of distributions might we consider the exponential distribution to belong?” or by way of more probing questions: “Can you provide an interpretation of what someone means when they say that the mean of the Cauchy distribution does not exist?”, for example. The session ended with each group giving a presentation of their findings. This served the dual purpose of both giving the students time to clarify their findings in their own minds and of allowing me to assess the extent of their learning.

In the following three sections we provide a distillation of the mathematics that took place in the classroom subsequent to the introductory demonstration, discussion, consolidation, learning and research.

3. The potential candidates

The students were asked to consider the suitability of several well-known continuous distributions with regard to modeling the longevity of a torch, and subsequently to nominate the one they deemed most appropriate. In this section the candidate distributions are grouped according to their families; for example, the exponential distribution is actually a special case of the gamma distribution and is thus considered to be in that family. We denote, for a given family of distributions (indexed by some integer m), the value of q such that $P(q < X < kq)$ is maximized by $q_m(k)$. The corresponding maximum value of $P(q < X < kq)$ is given by $A_m(k)$.

Whilst it was feasible for all of the students to consider the aptness of each of the distributions discussed here, it was clearly not possible, in a single session, for each of them to calculate both $q_m(k)$ and $A_m(k)$ for each family. Besides, the focus of the workshop was on acquiring statistical knowledge and understanding rather than on developing technical skill with regard to calculus; indeed, our particular problem served merely as a contextual, and hence motivational, element to the workshop. Thus, in order

to be able to achieve the aims stated in the previous section, each group was given responsibility for the calculation of $q_m(k)$ and $A_m(k)$ for just two families of distributions. These results were then shared amongst the groups towards the end of the session.

I found it useful both to prompt students at appropriate points and to ask probing questions, thereby getting them to think a little more deeply about the characteristics and the suitability, with respect to our scenario, of each of the distributions. Therefore, at the end of some of the following subsections, questions are provided that might be put to students in this regard.

The normal distribution

The random variable $X \sim N(\mu, \sigma^2)$, of which $Z \sim N(0,1)$ is a special case, has pdf given by

$$f_1(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\}, \quad -\infty < x < \infty.$$

On using (1) we find, after some manipulation, that

$$q_1(k) = \frac{\mu}{k+1} + \sqrt{\left(\frac{\mu}{k+1}\right)^2 + \frac{2\sigma^2 \log k}{k^2 - 1}}.$$

The normal distribution might be regarded as one of the core elements of statistics. It is commonly used to model the distribution of measurement errors in experimental results or the variation of component dimensions in manufacturing processes. Not only is the normal distribution used extensively in modeling, but it also forms the basis of much statistical theory. It was one of the first distributions to acquire a formal definition, originating from the work of Abraham de Moivre in the early eighteenth century. Under certain conditions, the normal distribution can also be used as an approximation for other distributions; even discrete ones, such as the binomial or Poisson. It is a continuous [probability distribution](#) describing data that clusters around some [mean](#) value in a symmetric fashion. The graph of its pdf is bell-shaped, with the peak at the mean.

Questions:

Might this provide a realistic model to use in the situation we are currently considering?

Can you think of a reason why some might not deem it suitable for modeling the longevity of a torch?

Why, in practice, would the fact that X can take negative values not cause any problems?

Gamma distribution

Next, if $X \sim \Gamma(\lambda, t)$ then it has pdf

$$f_2(x) = \frac{1}{\Gamma(t)} \lambda^t x^{t-1} e^{-\lambda x}, \quad \lambda, t > 0 \text{ and } x \geq 0,$$

where $\Gamma(t)$ is the gamma function defined by $\Gamma(t) = \int_0^\infty x^{t-1} e^{-x} dx$. Using (1) we obtain

$$q_2(k) = \frac{t \log k}{\lambda(k-1)}.$$

For the particular case in which $t = 2$ (see [Figure 1](#), where $\lambda = 1$) we find, after a considerable amount of simplification, that

$$A_2(k) = \left(\frac{1}{k}\right)^{\frac{2k}{k-1}} (k^2 + 2k \log k - 1).$$

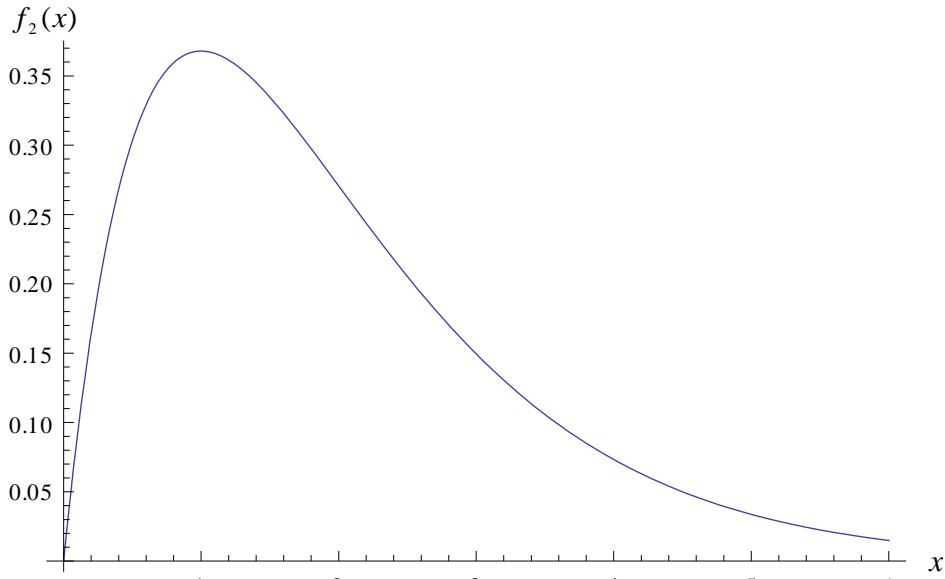


Figure 1: The gamma distribution with $t = 2$ and $\lambda = 1$.

The particular case for which $t = 1$ results in the *exponential distribution* (see [Figure 2](#), where $\lambda = 1$ once more), given by

$$f_2(x) = \lambda e^{-\lambda x}, \quad \lambda > 0 \text{ and } x \geq 0.$$

We have

$$A_2(k) = \frac{k-1}{k} \left(\frac{1}{k}\right)^{\frac{1}{k-1}}.$$

This distribution is used to model times between events that occur randomly in time, but at a constant average rate. As we might expect from an intuitive point of view,

$$A_2(k) = \frac{k-1}{k} \left(\frac{1}{k}\right)^{\frac{1}{k-1}} \rightarrow 1 \text{ as } k \rightarrow \infty.$$

We can think of λ and t as determining the ‘scale’ and ‘shape’ respectively of the gamma distribution. Although the parameter t can take non-integer values, when it is a positive integer the resultant distribution describes the sum of t exponential random variables, each independently and identically distributed with mean λ . Note also that special cases of the gamma distribution lead to another distribution students may be familiar with; the *chi-squared distribution*.

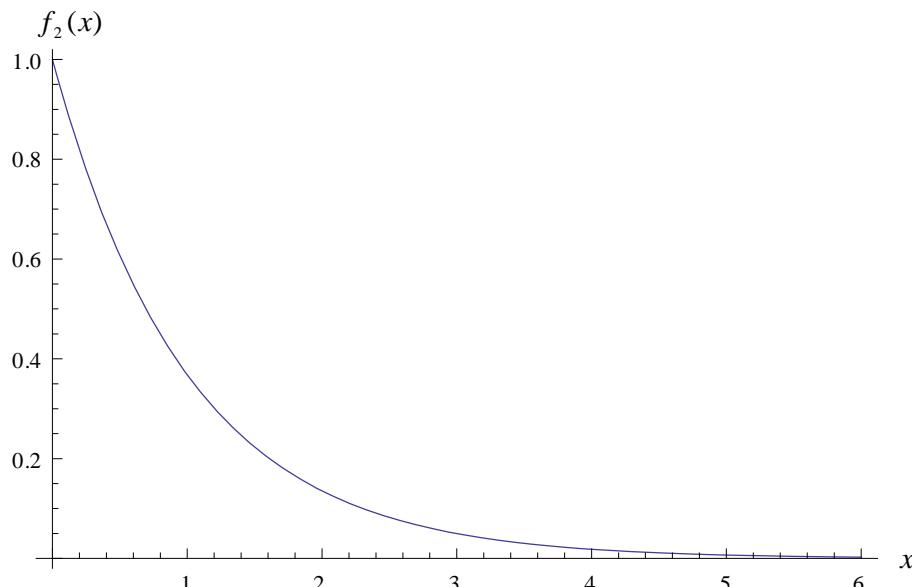


Figure 2: The exponential distribution (the gamma distribution for the case $t=1$).

The gamma distribution has, as a consequence of the fact that it may in certain cases be regarded as the sum of one or more exponentially distributed variables, applications associated with intervals between events. Examples of its use include queuing models, the flow of items through manufacturing and distribution processes, and the load on web servers and the many and varied forms of telephone exchanges. Also, owing to its moderately skewed profile, it can be used as a probability model in a range of disciplines, including climatology, where it is a workable model for rainfall, and financial services, where it has been used for modeling insurance claims and the size of loan defaults.

A good example demonstrating the application of the gamma distribution to a real-life situation is given in [Jones \(2009\)](#). The author considers the problem of modeling the time it takes to recruit patients into a clinical trial. This is a very important consideration in any clinical trial since it may have a major impact on whether or not the drug development program completes on time. In the model it is assumed that the interval of time between recruiting one patient and the next has an exponential distribution; bearing in mind the opening comment in the previous paragraph, the connection with the gamma distribution is now obvious. In the aforementioned article it is explained how simple statistical models involving the gamma distribution can be used to predict the time to complete recruitment.

Incidentally, the exponential distribution is rather special in that it is the only continuous distribution possessing the so-called *memoryless* property; see [Griffiths \(2006\)](#) for an explanation. This is something else that is certainly worth exploring with the students.

Questions:

For the particular case in which $t = 2$, note that $A_2(k)$ is independent of the parameter λ . Can you provide a simple mathematical explanation of this fact?

Is this also true for other values of t ?

What exactly do we mean by the ‘memoryless property’?

Beta distribution

We now consider $X \sim \beta(a, b)$ with pdf

$$f_3(x) = \frac{1}{B(a,b)} x^{a-1} (1-x)^{b-1}, \quad 0 \leq x \leq 1,$$

where

$$B(a,b) = \int_0^1 x^{a-1} (1-x)^{b-1} dx = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$$

is the *beta function*. Although this distribution is generally defined for $a, b > 0$, we assume here that $a, b > 1$.

By way of (1) we obtain, after some manipulation,

$$q_3(k) = \frac{k^{\frac{a}{b-1}} - 1}{k^{\frac{a+b-1}{b-1}} - 1}.$$

For the case where $a \geq 2$ is an integer and $b = 2$ it is possible to show that

$$A_3(2) = \frac{(2^a - 1)^{a+1}}{(2^{a+1} - 1)^a}.$$

Note that here $q_3(k) \rightarrow \frac{1}{2}$ and $A_3(2) \rightarrow 1$ as $a \rightarrow \infty$. We can see these results in the light of the graphs of our distribution as a increases. The shapes of the distributions for $a = 2$, $a = 3$ and $a = 6$ are illustrated in [Figures 3, 4](#) and [5](#) respectively.

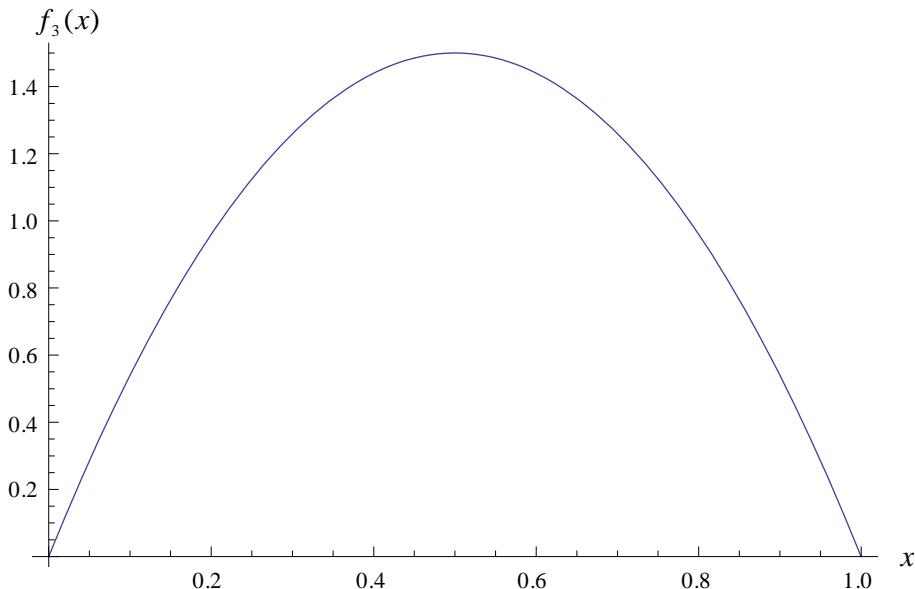


Figure 3: The beta distribution with $a = 2$ and $b = 2$.

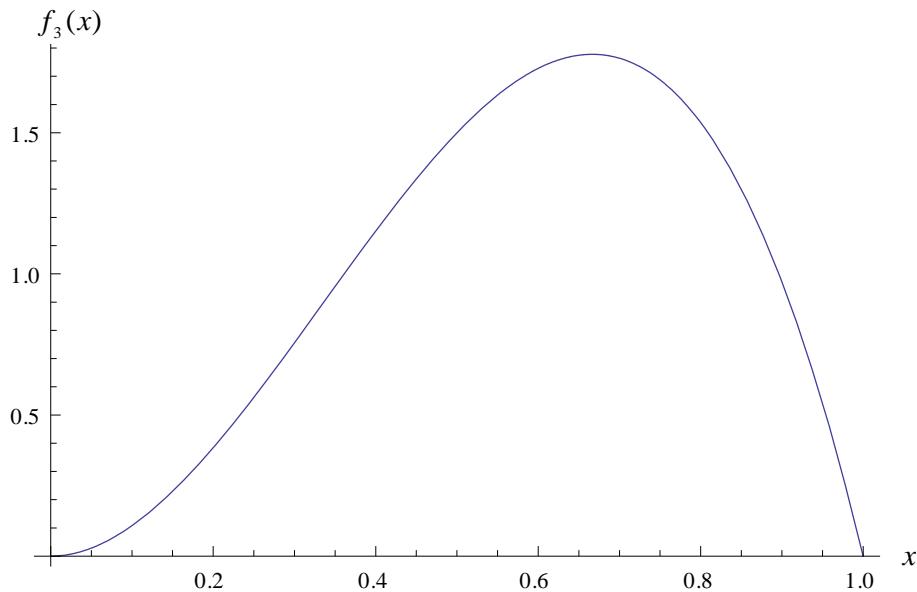


Figure 4: The beta distribution with $a = 3$ and $b = 2$.

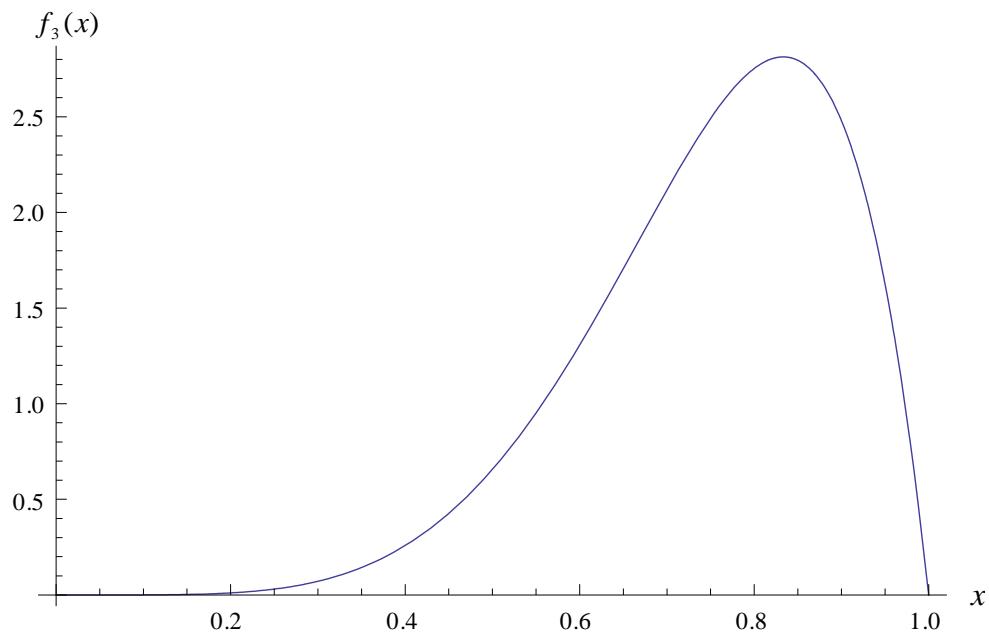


Figure 5: The beta distribution with $a = 6$ and $b = 2$.

The special case of the beta distribution for which $a = b = 1$ gives us the *continuous uniform distribution* with pdf $f_3(x) = 1$, $0 \leq x \leq 1$. It is clear now that

$$q_3(k) = \frac{1}{k} \text{ and } A_3(k) = \frac{k-1}{k}.$$

The beta distribution models events which are constrained to take place within an interval defined by a minimum and maximum value. For this reason, it is sometimes used in project planning to describe the time to completion of a task.

Question:

Do the above facts indicate that the beta distribution is suitable for our purposes, or otherwise?

Cauchy distribution

In this case X has the following pdf:

$$f_4(x) = \frac{1}{\pi(1+x^2)}, \quad -\infty < x < \infty,$$

as can be seen in [Figure 6](#). This actually arises on considering the random variable $X = \tan U$ where U has the continuous uniform distribution on the interval $[-\pi/2, \pi/2]$. We may easily show that

$q_4(k) = 1/\sqrt{k}$, from which it follows that

$$A_4(k) = \int_{1/\sqrt{k}}^{\sqrt{k}} \frac{1}{\pi(1+x^2)} dx = \frac{1}{\pi} [\tan^{-1} x]_{1/\sqrt{k}}^{\sqrt{k}} = \frac{2 \tan^{-1} \sqrt{k}}{\pi} - \frac{1}{2}.$$

So, for example,

$$A_4(3) = \frac{2 \tan^{-1} \sqrt{3}}{\pi} - \frac{1}{2} = \frac{1}{6}.$$

The Cauchy distribution is named after the nineteenth mathematician [Augustin Cauchy](#). It is both symmetrical and heavy-tailed. Heavy-tailed means that a high proportion of the population is comprised of extreme values. In fact, this distribution is so heavy-tailed that its mean does not exist! It turns out there is no analytical definition of moment-based properties such as the mean or variance. The Cauchy distribution does, however, possess a median, and this can serve as the location parameter.

An application of the Cauchy distribution occurs in software testing, where it is necessary to use datasets containing a few extreme values that could potentially trigger some adverse reaction. This distribution is clearly not suitable, in its present form at least, for modeling the longevity of a torch.

Questions:

Might we consider this a feasible probability model if it were translated so that the median was positive? If not, then what reason could you give?

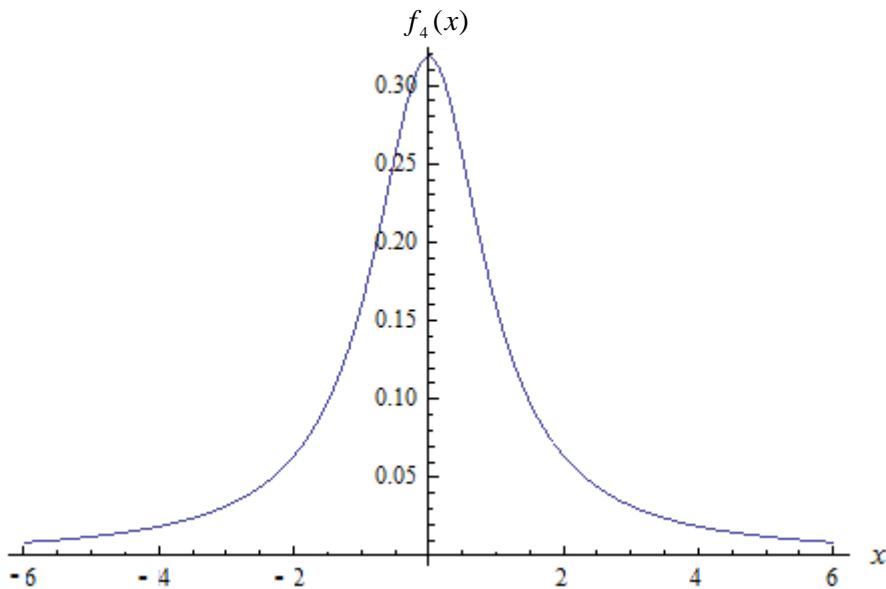


Figure 6: The Cauchy distribution.

Weibull distribution

This distribution has pdf

$$f_5(x) = x^{\beta-1} \alpha \beta \exp(-\alpha x^\beta), \text{ with } \alpha, \beta > 0 \text{ and } x \geq 0,$$

noting that this reduces to the exponential distribution when $\beta = 1$.

We find here that

$$q_5(k) = \left(\frac{\beta \log k}{\alpha(k^\beta - 1)} \right)^{\frac{1}{\beta}} \text{ and } A_5(k) = \frac{k^\beta - 1}{k^\beta} \left(\frac{1}{k} \right)^{\frac{\beta}{k^\beta - 1}}.$$

This distribution, named after the Swedish engineer Wallodi Weibull, is often used to model time to failure of manufactured items and has become one of the principal tools of reliability engineering and survival analysis. Its applications have in fact expanded somewhat, and now include finance and climatology. [Figure 7](#) shows an example of a Weibull distribution.

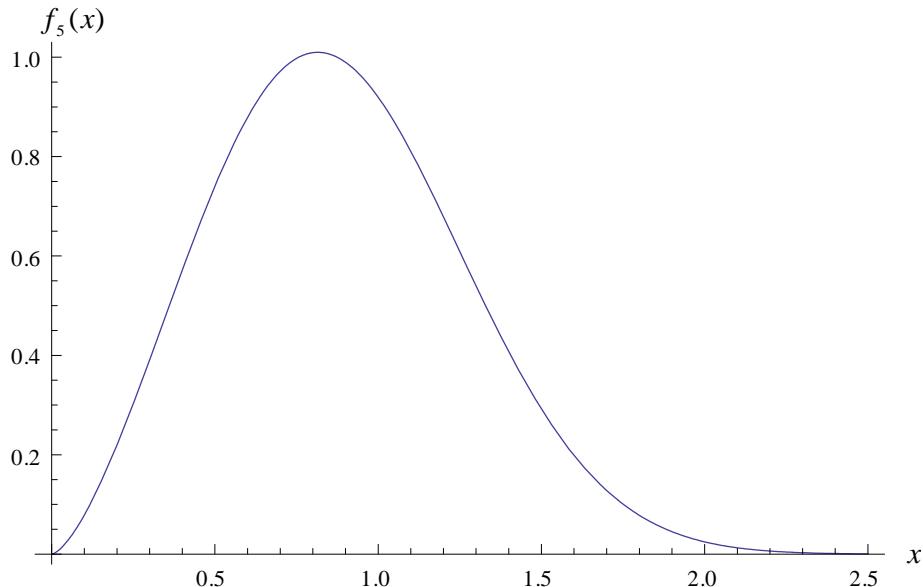


Figure 7: The Weibull distribution with $\alpha = 1$ and $\beta = 5/2$.

Questions:

From the point of view of mathematical transformations of functions, what roles do the two parameters α and β play here?

What features of this distribution might indicate that it is indeed a potentially strong candidate?

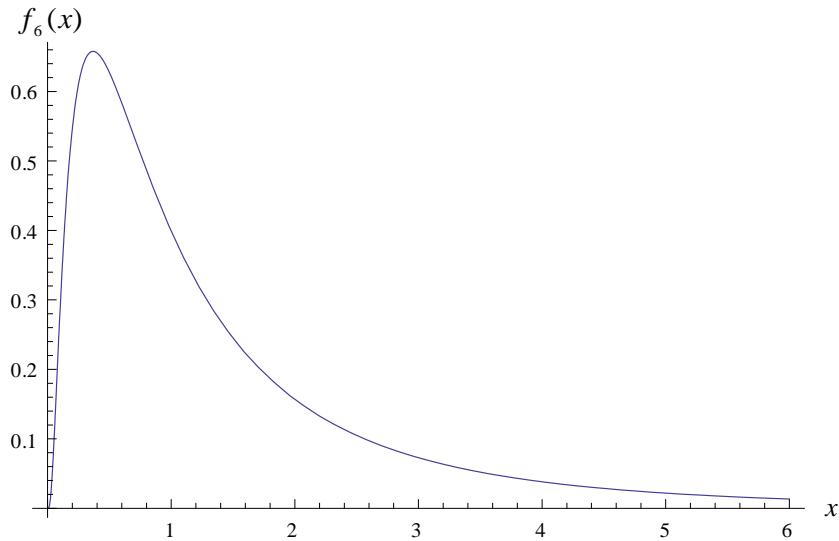
Log-normal distribution

The final distribution we consider here has pdf

$$f_6(x) = \frac{1}{x\sqrt{2\pi}} \exp\left\{-\frac{1}{2}(\log x)^2\right\}, \quad x > 0.$$

It arises from the standard normal distribution $Z \sim N(0,1)$ on considering the random variable $X = e^Z$.

This distribution is depicted in [Figure 8](#). The log-normal distribution describes many naturally occurring populations. In the mining and extraction industries it has been observed that where the value of an item is proportional to size, the population is probably log-normally distributed, with few valuable items and many non-commercial ones. An application of this distribution with regard to the purchasing power of earnings worldwide appeared in [Cooper \(2009\)](#). It is also studied in a recent JSE article; see [Olsson \(2005\)](#).

**Figure 8:** The log-normal distribution.

On using (1) we see that our probability is maximized when

$$\frac{k}{kx\sqrt{2\pi}} \exp\left\{-\frac{1}{2}(\log kx)^2\right\} - \frac{1}{x\sqrt{2\pi}} \exp\left\{-\frac{1}{2}(\log x)^2\right\} = 0.$$

This gives $(\log kx)^2 = (\log x)^2$, and hence $q_6(k) = 1/\sqrt{k}$. Note that this is the same as that for the Cauchy distribution.

4. Further results on limiting values

As we would expect, $A(k) \rightarrow 1$ as $k \rightarrow \infty$ for each of the distributions whose domain is either $x > 0$ or $x \geq 0$. However, it is more interesting to consider what happens to $q(k)$ as $k \rightarrow 1^+$. Let us denote this limiting value by $q(1^+)$. If k is close to 1 then, for a continuous random variable X with pdf $f(x)$, we have that

$$P(q < X < kq) \approx f(q)(kq - q) = qf(q)(k - 1),$$

from which we see that in order to calculate $q(1^+)$ we need to maximize $xf(x)$ with respect to x . For example, considering the standard normal distribution:

$$\frac{d}{dx} \{xf_1(x)\} = \frac{d}{dx} \left\{ \frac{x}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}x^2\right) \right\} = \frac{1}{\sqrt{2\pi}} (1 - x^2) \exp\left(-\frac{1}{2}x^2\right),$$

from which we obtain $q_1(1^+) = 1$. This really is rather interesting as the pdf of $X \sim N(0,1)$ has a point of inflection at $x = 1$. This ‘coincidence’ has occurred since $df_1/dx = -xf_1(x)$ so that

$$\frac{d^2 f_1}{dx^2} = \frac{d}{dx} \left(\frac{df_1}{dx} \right) = \frac{d}{dx} \{ -xf_1(x) \}.$$

In fact, as is easily checked, such a coincidence also occurs for the Cauchy distribution, defined via $f_4(x)$. To show that this will not generally be the case we just need to consider the gamma distribution, given by $f_2(x)$. Here we find that for $1 < t \leq 2$ there is a point of inflection at $x = \frac{1}{\lambda}(t-1+\sqrt{t-1})$ while for $t > 2$ there are two points of inflection, one at $x = \frac{1}{\lambda}(t-1+\sqrt{t-1})$ with the other being at $x = \frac{1}{\lambda}(t-1-\sqrt{t-1})$. It is also true that $q_2(1^+) = t/\lambda$, from which it follows that the only occasion on which $q_2(1^+)$ coincides with a point of inflection is when $t = 2$.

For the Weibull distribution we find that

$$q_5(k) = \left(\frac{\beta \log k}{\alpha(k^\beta - 1)} \right)^{\frac{1}{\beta}} \rightarrow \left(\frac{1}{\alpha} \right)^{\frac{1}{\beta}} \text{ as } k \rightarrow 1^+,$$

while the points of inflection are given by

$$\left(\frac{3(\beta-1) \pm \sqrt{(5\beta-1)(\beta-1)}}{2\alpha\beta} \right)^{\frac{1}{\beta}},$$

for values of α and β such that this expression gives positive real numbers.

5. The issue of uniqueness

To this point all our distributions have possessed unique values of $q(k)$ for each $k > 1$. However, this will not necessarily always be the case. The random variable X with pdf $f_7(x) = 1/kx$, $1 \leq x \leq e^k$, has $P(q < X < kq) = \frac{1}{k} \log k$ for $1 \leq q \leq e^k/k$. It is also possible to construct pdfs that have exactly two distinct values of $q(k)$ for some $k > 1$. Indeed, this makes an interesting challenge for the students. For example, the random variable X with the rather contrived piecewise pdf given by

$$f_8(x) = \begin{cases} \frac{2}{3} & 0 \leq x \leq 1 \\ \frac{2}{3}(x-1) & 1 < x \leq 2 \end{cases}$$

has $P(\frac{1}{2} < X < 1) = P(1 < X < 2) = \frac{1}{3}$ while all other probabilities of the form $P(q < X < 2q)$ are less than $\frac{1}{3}$.

6. Points for discussion

The first point to be made here is that this is very much a tried-and-tested activity; one that does seem to engage the participants, helping them to appreciate the fact that there is a little more to the statistical process than is sometimes portrayed in textbooks. The last time I ran such a session, I was struck by the number of students saying that they had enjoyed participating simply because of the fact that these

potentially abstract notions had been placed in some sort of context for them. Indeed, the importance of presenting new theoretical ideas in context is a recurring theme in [Howley \(2008\)](#). The workshop setting also allows for both collaborative and independent learning to take place. Furthermore, although this activity is very much based on the investigation of pdfs of continuous random variables, it does give bright students plenty of scope to explore more advanced areas of calculus, the idea of a limit, special functions and even parameter estimation.

It has to be admitted, however, that thus far this has only been carried out with relatively small classes of bright undergraduate students. Although the workshop is potentially highly flexible, it is inevitable that some of this flexibility will be lost in situations where classes are particularly large or contain students with wide-ranging abilities and experiences. Another factor that needs to be taken into account is the amount of time available. If time was at a premium, as was the case here, then each group may consider in detail just one or two of the aforementioned distributions. This scenario presents an excellent opportunity for peer-teaching to take place, as espoused in [Zacharopoulou \(2006\)](#). Indeed, our session ended with each group providing an account of their solution to the problem for the particular distributions they had studied, a coherent argument for or against the use of these distributions to model the longevity of a torch and, with justification, a nomination for the most suitable distribution. There is also the possibility for students to continue the work at home. In this case, matters could be tied up in a brief follow-up session.

The final presentations proved to be extremely valuable in terms of the discussion that was generated. When considering which of the distributions might be deemed suitable, students soon appreciated, under questioning from their peers, the need to be clear about the reasons influencing their choice. Some of the points raised were: Should we discount distributions taking negative values? Do we believe that the distribution should be skewed rather than symmetric? Should our random variable be able to take arbitrarily large values? Is anyone of the opinion that none of these distributions would be suitable? Some also considered the problem of devising a practical experiment to determine whether or not a particular distribution is actually a genuine candidate. This of course also raises the issue of parameter estimation.

The general consensus was that the Weibull distribution would, for some appropriate selection of parameter values, be the model of choice for our scenario. Indeed, experience tends to show that failure data can often be well modeled by this distribution, and, for the modeling assumptions stated in Section 2, it might thus be deemed suitable. For the Weibull distribution, β is known as the ‘shape’ parameter. If $\beta = 1$ then, as already mentioned, we obtain an exponential distribution. This leads to a failure rate that remains constant over time (recall that the exponential distribution has the ‘memoryless’ property). When $\beta < 1$ the failure rate decreases as time goes by. The more plausible situation for the lifetime of a torch, however, is that the failure rate increases with time, corresponding to $\beta > 1$. This would occur if the torch underwent some sort of ageing process, and was more likely to die as time elapsed. It might be argued that this is in fact the case since the filaments of light bulbs both evaporate over time and undergo a process known as *creep*. For further general information about the Weibull distribution, visit the [Wikipedia website \(2010\)](#).

With the first aim of the workshop having been achieved, the students went on to consider the problem of calculating the length of the walk such that the expected number of torches dying out on the return journey is maximized. This involved evaluating $q_5(2)$ for appropriate values of the scale and shape parameters, α and β respectively. As discussed above, the students decided it would be sensible to choose $\beta > 1$. There remained the task of choosing an appropriate value of α .

Of course, in reality things might not be quite so simple. Indeed, the removal of one or more of the initial assumptions stated in Section 2 would give rise to some challenging modeling problems. For example, rather than assume the longevity of a torch, as a single unit, possesses some well-known continuous distribution, we might take account of the fact that the lifetimes of bulbs and batteries would each follow their own distributions. Then, if X and Y are random variables representing the lifetimes of bulbs and batteries respectively, the longevity of the torch is given by $W = \min[X, Y]$. It is quite possible that the distributions of X and Y belong to different families. So, for example, the lifetime of a bulb may follow a Weibull distribution, while that of a battery might best be modeled by way of a normal distribution. Further complications might arise if X and Y are dependent as random variables. If the batteries are dying out then the bulb will probably be fairly dim, thereby lessening its aging process. These ideas would provide extension material for exceptionally talented students.

There is also much scope for computational work here. In order to graph the various pdfs or carry out appropriate numerical work then [Mathematica \(2007\)](#) and [R \(2004\)](#) provide plenty of options. Our problem also provides an ideal opportunity to conduct Monte-Carlo simulations.

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