Pinochle Poker: An Activity for Counting and Probability

Jacqueline Wroughton
Joseph Nolan
Northern Kentucky University

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Abstract

Understanding counting rules is challenging for students; in particular, they struggle with determining when and how to implement combinations, permutations, and the multiplication rule as tools for counting large sets and computing probability. We present an activity – using ideas from the games of poker and pinochle – designed to help students solidify and expand upon counting techniques while also promoting critical thinking in the classroom. While this activity has been used in college level courses, we believe it would also be applicable in a high school discrete mathematics class or in any probability course having substantial emphasis on these topics. We present and discuss the activity including desired learning outcomes, rationale, opportunities for teachable moments, and potential follow-up assignments.

1. Introduction

Elements of probability and statistics are taught across the educational spectrum: from elementary grades (Watson and Moritz, 2000) through the college level (Rumsey, 2002). Recent developments in curricula have substantially increased the frequency of appearance for these topics within the K-12 program (Common Core Standards, 2010). Therefore, future teachers require stronger preparation than ever.

Statistics educators have been encouraged by the GAISE report to increase student involvement in classroom activities. In particular it is recommended that activities do not lead students step by step but rather allow students to discuss and think critically about the problem at hand (Aliaga, Cuff, Garfield, Lock, Utts, and Witmer, 2005). In addition to recommendations from

GAISE, past studies contribute evidence in support of activity-based learning as an effective tool in teaching probability. For example, Shaughnessy (1977) presents findings regarding the use of activities to help dispel probabilistic misconceptions. An interesting experiment conducted by Umble and Umble (2004) showed the effectiveness of an active learning exercise to help students better understand conditional probability concepts related to the Monte Hall problem. In this paper we provide details of a poker-related classroom activity that aligns with the GAISE report to help our college students (many of whom are future educators) deepen their understanding of combinatorics and probability.

Why poker? The primary reason is motivation. Many of our students play poker from time to time; this can be helpful to garner their interest in the activity. Most are already familiar with poker and the rankings of the different hand-types. While only a few have ever calculated any probabilities related to poker, they generally have some intuitive understanding that the rankings of hand-types are based on the likelihoods of achieving those hands.

Why a pinochle deck? Poker with a standard 52-card deck is fairly well known and certainly well studied; readers wishing to learn more about poker in general may be interested in the following web-links:

- http://www.pagat.com/poker/rules/ (Rules of Poker)
- http://people.math.sfu.ca/~alspach/computations.html (Alspach's Mathematics & Poker Page)

One of the intriguing aspects of this activity is the added complexity of calculation brought about by changes to the deck. The pinochle deck represents a very different set of cards. While it maintains the four standard suits (clubs, diamonds, hearts, and spades), it contains only six denominations (ace, king, queen, jack, ten, and nine) and additionally there are two of each possible card! This duplication of cards creates some interesting twists in the probability calculations when we consider drawing poker hands from this deck. The Pinochle Poker Activity invites students to explore these calculations as they strive to determine the appropriate ranking of hand-types for this deck. While this activity has been tailored for use with mathematics and statistics majors as well as future teachers at Northern Kentucky University (NKU), we believe the activity would be both entertaining and applicable to other student demographics as well. For example, with minor adjustments, the activity might be directly applied to a discrete mathematics class within the secondary schools.

2. The Pinochle Poker Activity

Counting techniques such as combinations and permutations are important for the purpose of accurately counting possible outcomes without the need to list each individual possibility. In our college classrooms, we have often watched as students attempt to imitate these techniques without demonstrating a complete understanding of their calculation. Quite frequently we see that critical thinking skills are underused. For example, many students prefer to have a "formula" for everything, but that isn't generally reasonable for counting large sets of outcomes. In light of these points, we have developed this activity to be fun for our students while at the same time emphasizing the following primary goal of having students – through careful, critical thinking – develop a deeper understanding of probability concepts and combinatorial techniques. The activity itself consists of three parts:

1. An introduction which involves students in counting the number of ways to create each possible poker hand using a standard 52-card deck. This helps to level the playing field – getting students on the same page with their understanding of poker hand-types while at the same time demonstrating the process of counting. The amount of assistance provided by the instructor should be appropriate to the level of the students.

2. For the main activity, students generally work in pairs to calculate hand-counts and probabilities for the pinochle deck. They can make some use of earlier calculations but must think very carefully about the repercussions of having duplicate cards.

3. The third part asks students to develop a report comparing the rankings of poker hand-types using the two different decks of cards. More advanced students may also be asked to generalize the methods to other arbitrarily chosen scenarios.

Sections 2.1-2.3 describe the three parts of the activity in greater detail and provide suggestions for the instructor to consider in supervising the activity. Materials for the activity (classroom handouts) are available in Appendix A. These may of course be adjusted at the discretion of the instructor and according to the level of the students. Detailed solutions for all hand-types are available in Appendix B. These also include instructional comments and useful suggestions to help instructors in conducting the activity.

2.1 Regular Poker

The first part of the activity is designed to assist students in developing a general process by which to formulate the number of each hand-type possible using a standard deck of cards. Initially students must recognize that the hand-types are different; therefore, the process of calculation must remain fluid. There is no "one simple formula" approach. In general we have found that a fair amount of instructor and/or full-class involvement is appropriate at this stage. The extent may depend on the level of the students. For example, in a class for middle grades educators, we go through each possible hand together on the chalkboard. The students do most of the work, while the instructor uses guided questioning techniques to help illustrate the thought process involved in calculations (King and Rosenshine, 1994). Alternatively, in a class for statistics majors, the instructor might do only one or two examples in this way and allow the students to complete the rest in groups.

For each hand students should recognize the need to specify five cards – each card consisting of a denomination and a suit. To some extent, it is appropriate to consider these two components separately (and then use the multiplication rule). Within a poker hand we must also remember that the order in which cards are dealt is not relevant. We provide a full discussion of the classroom discussion process for each hand in Appendix B. Below is an example of how the demonstration might go for two-pair.

If a hand contains two 7's, two 8's, and one queen, the hand is "two-pair, 8's and 7's". It has the same hand-value whether it is dealt in the order 8877Q or 877Q8 or any other order. To specify the hand, we'll need to specify the denominations for each pair (i.e. the 8's and the 7's) as well as the single; and then specify suits for each card (since for example holding 7♠ and 7♦ is different from holding 7♥ and 7♣).

- *Specify Denominations*. We must ultimately select three different denominations (two for pairs and one more for the single). Students will likely need guidance in assessing whether the order in which these denominations are selected is important. In fact, it is only partially important. There are C(13,2) choices for denominations of our pairs (it doesn't matter whether the 7's or 8's come first), but then conditionally 11 choices for the denomination of the single (it does matter that the Q is alone, and not one of the pairs). Observant students might note that $C(13,2)\times 11 = 13\times C(12,2)$ as we could just as easily decide the denomination of the single first.

- *Specify Suits*. For each pair we must take two of four suits, while for the unmatched card we must pick only one. Note that in no case does the order of selection for suits make a difference. However, if we select hearts-clubs and diamonds-clubs for the pairs, it does make a difference which pair is associated to which group of suits. Thus the multiplication rule will apply: $C(4,2)\times C(4,2)\times C(4,1)$. This might be thought of as the process of choosing the suits for the highest pair, choosing the suits for the lowest pair, and then choosing the suit for the single.

- *Put it together*. For every way there is to specify the denominations, there are $C(4,2)\times C(4,2)\times C(4,1)$ ways to specify the suits. Therefore, again applying the multiplication rule, there are $C(13,2)\times 11\times C(4,2)\times C(4,2)\times C(4,1) = 123,552$ different hands at the rank of two-pair.

As previously mentioned, a summary of all pertinent calculations for the standard deck and key points on which an instructor might focus can be found in Appendix B. After having completed the first part of the activity, students should be much more confident in their understanding of the standard deck of cards and the constitution of each possible hand. Having completed the counting calculations for this deck, they should also have substantial insight into the reason we apply the different counting rules (combinations, permutations, the multiplication principle) at different stages. One might close this portion of the activity with a discussion of differences between hand calculations. For example, the class could contrast the calculations for two-pair, three-of-a-kind, and a full-house. In each of these calculations, when selecting denominations order matters at times, and at other times it does not. Another point of discussion is to help students recognize situations where subsets must be subtracted. For example, a straight flush would be initially counted amongst all flushes.

2.2 Pinochle Poker

Having calculated probabilities for the different poker hand types using a regular deck of playing cards, students have gained experience (and hopefully some confidence) implementing combinations, permutations, the multiplication rule, and ideas of subsets. We begin the second part of the activity with a discussion of how changes to the deck might affect the rankings of the hands. Some questions to consider:

- What would happen if we added (or removed) suits?
- What would happen if we added (or removed) denominations?

We specifically leave out the question "what would happen if we duplicated cards?" because we want them to discover that aspect of the pinochle deck for themselves.

After discussion, students working in groups of two or three are given a pinochle deck and the task of identifying and counting all possible poker hands using this deck. The pinochle deck is generally unfamiliar to most students, so their first task is really to identify its properties. As previously mentioned it consists of 48 cards with four suits (clubs, diamonds, hearts and spades) and six denominations within each suit (nine, ten, jack, queen, king, ace). While the reduced number of denominations would seem to make counting easier, the fact that the deck contains duplication (there are two of every card – i.e. two kings of clubs, two queens of diamonds, etc.) serves to make counting more challenging. Additionally, they must consider other nuances such as:

- Five-of-a-kind is a "new" possible hand.
- It is now possible for hands containing one or two pair to also be flushes (e.g. AAKKQ all diamonds). Students (perhaps with some direction from the instructor) must determine how such a hand should be counted – and also recognize that it shouldn't be counted in two different groups.

Students are asked to find the number of ways to create the same types of hands as they had previously done with the standard deck (with the addition of five-of-a-kind). Again depending on the level of the students, leading questions from the instructor are often appropriate. To maintain the discovery format, it is preferable that the instructor float around from group to group. A summary of all pertinent calculations for the pinochle deck, including key points and tips for instructors is given in Appendix B so we will not reiterate those here. We would however like to highlight what we feel are some important instructional notes:

- Students should be asked to explain their counting methods. One of our key goals is for students to understand ***why*** they are using a particular combination or permutation.
- Students are asked to calculate the probability of each type of hand directly (i.e. they should not use the complement rule to come up with the last hand). They should then ensure that the probabilities correctly add to 1 (or equivalently their computed counts

total C(48,5) = 1,712,304) when they have completed the activity. Invariably when they check they will find that errors exist (often the sum is greater than 1 because they have not subtracted overlap). An ideal way to handle this is to provide the students with a specific hand (that has been counted twice) and ask them in what group they placed it. If a stronger hint is preferred, the instructor may wish to provide more direct assessment of which calculations need revision (Table 1 below summarizes the counts and probabilities for reference). The amount of aid to be provided should be appropriate to the level of the class.

- The biggest hurdle for students in these calculations is evidenced by the first one they typically tackle – the Royal Flush. Because there are 2 of each card (i.e. two aces of hearts, two kings of hearts, etc.) in the pinochle deck, students most often come up with an initial answer that there are 8 Royal Flushes available in the 48-card pinochle deck. This is of course not the correct answer – and because this hand is usually the first they attempt, it may be appropriate just to tell them that 8 is not correct, and have them spend a bit more time thinking about the problem before giving further hints. The best hint we have found is made easiest by having constructed pinochle decks that consist of both red-backed and blue-backed cards. After asking students to lay out the royal flushes in one suit (say spades), pick up the two aces of spades and switch them – pointing out that the royal flushes just created are different from the previous two, because different cards are used (the evidence of the red vs. blue backing is useful to convince them). This technique may be used with any other hand as well.

Table 1: Hand Counts and Probabilities

		Regular Deck		Pinochle Deck	
Hand	Example	Count	Probability	Count	Probability
Five-of-a-Kind	K♦ K♠ K♣ K♦ K♣			336	0.000196
Royal Flush	A♦ K♦ Q♦ J♦ T♦	4	0.00000154	128	0.0000748
Straight Flush	K♦ Q♦ J♦ T♦ 9♦	36	0.00001385	128	0.0000748
Four-of-a-Kind	A♠ A♥ A♣ A♦ 9♣	624	0.00024	16,800	0.00981
Full House	A♥ A♦ A♠ 9♥ 9♣	3,744	0.00144	47,040	0.02747
Flush					
Two-pair Flush	K♠ K♠ J♠ J♠ T♠			480	0.000280
One-pair Flush	T♠ T♠ A♠ Q♠ J♠			1,920	0.001121
High-card Flush	A♠ Q♠ J♠ T♠ 9♠	5,108	0.00197	512	0.000299
Straight	A♠ K♥ Q♥ J♦ T♥	10,200	0.00392	65,280	0.0381
Three-of-a-Kind	9♠ 9♦ 9♣ A♦ Q♣	54,912	0.02113	215,040	0.1256
Two-Pair	K♦ K♠ J♦ J♠ 9♠	123,552	0.04753	375,840	0.2195
One-Pair	A♠ A♥ Q♥ T♣ 9♥	1,098,240	0.4226	858,240	0.5012
High-Card	A♠ K♥ J♠ T♦ 9♣	1,302,540	0.5012	130,560	0.0762
Total		2,598,960		1,712,304	

2.3 Rankings Report and Other Assignments

To conclude the activity, students are required to develop a summary report comparing and contrasting poker rankings using the two decks of cards (higher ranked hands are those of the smallest probability). Their reports are generally expected to include at least the following:

- A comparison table of probabilities for use in comparing regular poker to pinochle poker. (Note that a table of counts is not appropriate because the total number of possible hands is not the same for both decks.) Students should remember that some additional hands (e.g. five-of-a-kind) are possible using the pinochle deck.

- A discussion of differences in rankings (in particular students should notice that a "high-card" hand ought to rank very highly when playing with a pinochle deck while it is the worst hand using a regular deck).

For more advanced students (in particular math/stat majors) the following items are also appropriate:

- Discussion of flushes. For a pinochle deck there are three different types (those that contain no pair, one pair, or two pair). How should these be ranked?

- Extension of these ideas into a deck that has *c* colors, *r* denominations, and *n* duplicates of each card. Advanced students should be able to fully develop an understanding of how the values of *c*, *r*, and *n* affect the calculations. The following are example challenge questions:

  - Build a deck in which a pair is a very strong hand (for example, stronger than flushes and straights).

  - Build a deck in which a straight is a very strong hand (for example, stronger than full-house or four-of-a-kind).

  - Construct a general formula to compute the probability of a full house using a deck of *c* colors, *r* denominations, and *n* duplications (one might use any other hand specification as well).

- Applications of the hypergeometric distribution. Some hands (e.g. flushes) represent applications of this distribution. Others represent extensions of ideas related to this distribution. One might ask students to consider the number of a specific suit (or a specific denomination) found in the hand.

- Expansion of probabilistic ideas into statistical methods. The selection of a poker hand is random sampling from a finite population. Analogous to an example considering minority baseball coaches utilized by Chance and Rossman (2006), students might consider the probabilities constructed for use in identifying potential cheating in a home poker game.

- Extension of these ideas into a deck that has wild cards. This is particularly challenging, since with wild cards one must often choose between hand-types when defining a particular hand. For example, if one holds the ace, king, queen, and jack of clubs along with a wild card – the hand could be any of the following: high-card, one-pair, straight, flush, or straight flush. These inconsistencies are discussed by Emert and Umbach (1996) and methods for handling them proposed by Lampe (2011).

Ultimately, the report portion of the assignment gives students time to reflect on the activity and more fully absorb what they have learned.

3. Closing Remarks

While at this point the Pinochle Poker activity has not been formally assessed, we believe that it brings three important components to our classroom. First, it engages students, who by nature are generally curious with regard to poker and seem to like its involvement in the activity. Second, this activity asks students to implement counting techniques and the rules of probability in ways that involve critical thinking. As teachers we see that as being of upmost importance. Lastly, the activity has something to offer a large range of student abilities. The general education student learns to compute basic probabilities, while more thought-provoking questions are available to more mathematically-inclined students.

This activity has been used primarily in two of our courses. The first is an introductory probability and statistics course required of all majors with a one-semester calculus prerequisite. The second is a basic probability course that is part of the program for middle grades educators who will teach mathematics. Overall, it appeared that the students did enjoy this challenge where the answers were not already known. Many began the activity under the impression that it would be easy since "everyone knows how hands are ranked." But once calculations for the pinochle deck were underway, you could see (and hear) them thinking, "what would happen if…" or "we didn't take into account…" When asked for informal feedback, students generally reported that the activity was more difficult than they anticipated. Since the students didn't already know what the answers should be, they had to take more time to consider all the possibilities (this was appreciated in particular by one student in a course evaluation comment).

We often observed that even students of strong mathematical ability tended to jump into an attempt at mathematical solution too quickly, without first putting thought into the process of constructing hands. While having the physical cards for demonstration is helpful in visualizing the construction of hands, we feel it is important that the activity forced students to stop and think about the problem and determine the appropriate questions and counting methods for themselves. Both correct as well as incorrect solutions are useful in the process of completing the activity as the instructor uses guided questions to direct the students' learning. The group discussion dynamic is also important as it acts as a catalyst for these questions; that is, it is a time that both the teacher and students can ask and answer questions. Ultimately we believe that students who not only come up with the correct answers to these questions, but can also explain them, are the ones that benefit most from this activity.

Appendix A

The worksheet on the following pages is used with our STA 110 Intro Probability course – the audience for this course is middle grades educators. Instructors interested in using this activity are welcome to modify the worksheet as may be appropriate to their audience.

STA 110 Pinochle Poker Activity

Materials needed: 1 standard 52-card deck of playing cards; 1 standard 48-card deck of pinochle cards; calculator, partner. Always remember that, as in any card game, partner is your friend!

Part I – As a class, we will calculate the probabilities for all of the possible poker hands using a regular deck (52 cards; labeled 2,3,4,5,6,7,8,9,T,J,Q,K,A in each of four suits – clubs, diamonds, hearts and spades). Here are the hand definitions (note that they are ranked from best hand to worst hand and in poker you always play the best if you have a choice):

1. Royal Flush: A K Q J T – all of the same suit
2. Straight Flush: 5 cards in the same suit in sequential order (including 5432A, but not AKQJT)
3. 4 of a kind: Having 4 cards of the same denomination and one card of a different denomination – such as 44448 or QQQQK
4. Full House: 3 of a kind + a pair (example: QQQ22) ranked by the three of a kind.
5. Flush: 5 cards of the same suit but not in a sequence (ranked by high card) – KJT52 of diamonds is a King-high flush
6. Straight: 5 cards in sequential order but not of same suit – Ace can be high as in AKQJT or low as in A2345.
7. 3 of a kind: 3 cards of the same denomination and two other cards of different denominations - for example 888A7. This hand is often known as “trips” or a “set”.
8. 2 pair: two different pairs and one card of a third denomination – ranked by the higher pair, e.g. QQ552.
9. Pair: two cards of the same denomination and three other cards of different denominations – for example QQJ52
10. High card: when you have no pair, no flush, no straight etc., hands are ranked by the highest card.

You will want to write down the calculations as we go through them as a class. As we gain more experience, I will be asking you to first attempt to calculate them without assistance.

Summary of probabilities (and number of hands)

Royal Flush: 0.00000154 (4)
Straight Flush: 0.00001385 (36)
Four-of-a-kind: 0.00024001 (624)
Full-House: 0.00144058 (3,744)
Flush: 0.0019654 (5,108)

Straight: 0.0039246 (10,200)
Three-of-a-kind: 0.021128 (54,912)
Two-Pair: 0.047534 (123,552)
Pair: 0.422569 (1,098,240)
High Card: 0.501177 (1,302,540)

Some Questions for Further Thought

- Why are the hands ranked in the order above?
- Which of the calculations did you find most difficult? Why?
- If we added/removed suits, what do you think would happen?
- If we added/removed denominations, what do you think would happen?

Part II: (You will do this in your groups of 2, mostly in class.) Suppose now that the deck of cards that you are playing poker with has changed – you are now using a pinochle deck (samples are available in class). **Your ultimate goal: Determine how poker hands should be ranked when playing 5-card poker with this deck!**

First go through the deck and note what types of cards are in this deck. You will only focus on the same types of hands as we did with the 52-card deck plus the added possibility of 5-of-a-kind!

- The starting point: What is different about this deck? What special considerations will you need to make in order to count poker hands using this deck (there are at least two!)?
- In order to rank the different hands, you will need to show how many ways to come up with each type of hand. Do it! And please make sure to show your calculations!
- Rank the hands for use in your write-up (below).
- Question for further thought: how does the addition/removal of duplicate cards affect the possibilities?

Part III: Your write-up for this activity (which you will complete as a group and turn in one copy) should have the following three components:

1. Compare and contrast the probabilities and rankings of the poker hands when using a standard deck vs. a pinochle deck. Provide tables of probabilities, and then in a paragraph discuss the results. Please also include all of your calculations from Part II in an appendix.
2. Which of the pinochle poker counts/probabilities did you find to be most difficult to calculate? In a well written paragraph, discuss the calculation and what you think made it hard to figure out.
3. You would like to create a deck of cards in which the full house is the highest ranking (i.e. least likely) hand. What do you think should be the features of this deck? Why?

Notes on write-up: Please use paragraphs, complete sentences, and reasonable grammar! Use tables in an appropriate manner; **show all calculations in an appendix**. Please submit one write-up per pair. You might decide to type the report, but the appendix will no doubt be hand-written.

Appendix B
Solutions and Commentary

The purpose of this appendix is to provide educators some insight into how students think about poker-hand probabilities. It is also designed to assist educators in facilitating the *Pinochle Poker* activity within their classrooms. In the activity, all regular-deck poker probabilities are computed first – and this is followed by computation for the pinochle deck. In this appendix, however, it was appropriate to discuss each hand for both decks, so that useful connections and extensions may be illuminated. Note that, within this appendix, *italicized* questions are questions that we would ask in class while leading the students through the activity.

B.0 General Comments

Before constructing hand counts, it is important that students understand that, to identify any poker hand, we must identify both the denominations and the suits for each of the five cards in the hand. It can be useful to diagram this for students as follows:

Denom - Suit	Denom - Suit	Denom - Suit	Denom - Suit	Denom - Suit
____________	____________	____________	____________	____________

As hands are counted, it is useful if students use this diagram to fill in **an example** hand as they count each part. For example, when the denomination for 3-of-a-kind is selected (but no suits have been picked – nor have the remaining two cards), the student's diagram might look like this:

5	5	5		
Denom - Suit	Denom - Suit	Denom - Suit	Denom - Suit	Denom - Suit
____________	____________	____________	____________	____________

Note that it does not matter what numbers or suits they fill in to their diagram – the purpose of the diagram is just to keep track of what has already been chosen (and what is left to choose). Students should also be reminded that the order in which they write the example cards is irrelevant.

Another notable point is that, in general, if two hands are of the same type at the poker table – they are ranked based on the highest cards (starting with A, K, Q...). In computing poker probabilities in this activity, we are concerned only with the types of hands. We make no allowance to differentiate between hands of the same type.

B.1 Royal Flush

<u>Definition</u>: A Royal Flush is defined as the ace, king, queen, jack, and ten (AKQJT) all in the same suit.

Regular Deck (thought process and calculations)

Students will easily recognize that there are only 4 possible Royal Flushes in a regular 52-card deck. In fact, this is so readily apparent to them that, especially in lower level classes, they will be unable to correctly count the number of royal flushes in a pinochle deck on their first attempt! In order to assist them with that component, it is appropriate to go through the thinking process carefully with students:

- *We need to specify the denominations on the cards. How many ways are there to do this?* Students should quickly recognize there is only one way – you must have exactly the five denominations defined.
- *We need to specify the suits for each of the five denominations. How many ways can this be done?* There are four suits in the deck.
- *How do we put these pieces together? Generally, our choices are addition, subtraction or multiplication.* In this case, the multiplication rule may be applied since the denominations and suit are independent. In other cases, the conditional multiplication rule will be used.

Based on these steps, the number of possible royal flushes available in a regular deck is $1 \times 4 = 4$.

Pinochle Deck (thought process and calculations)

Because there are 2 of each card (i.e. two aces of hearts, two kings of hearts, etc.) in the pinochle deck, students most often come up with an initial answer that there are 8 Royal Flushes available in the 48-card pinochle deck. This is not the correct answer – and because this hand is usually the first they attempt, it may be appropriate just to tell them that 8 is not correct, and have them spend a bit more time thinking about the problem before giving further hints.

The best hint we have found is made easiest by having constructed pinochle decks that consist of both red-backed and blue-backed cards. After asking students to lay out the royal flushes in one suit (say spades), pick up the two aces of spades and switch them – pointing out that the royal flushes just created are different from the previous two, because different cards are used (the evidence of the red vs. blue backing is useful to convince them).

Ultimately, the thinking process to construct royal flushes in the pinochle deck is as follows (ideally, students will figure this out for themselves, with some minimal guidance from the instructor):

- *Specify Denominations.* These are still effectively already specified by the definition of the hand – you must have AKQJT (so there is only one way).
- *Specify Suit.* Again there are 4 possible suits.
- *Recognize at this point you have selected 10 cards. For example perhaps you have the AAKKQQJJTT of spades.* To reduce this to five cards, you must choose one of the two aces, one of the two kings, one of the two queens, etc. There are 2 choices available for each of the 5 denominations.

Applying the multiplication rule to the above we now recognize that in a pinochle deck, there are $1 \times 4 \times 2^5 = 128$ Royal Flushes.

B.2 Straight Flush

Definition: A Straight Flush is defined as five cards in sequential order and all of the same suit that is not a Royal Flush. The hands 23456♥ or 89TJQ♣ are both examples of straight flushes. While technically in a poker game the Q-high straight flush beats the 6-high straight flush, for the purposes of this activity we do not differentiate among straight flushes in that way.

Regular Deck (thought process and calculations)

When considering how to count the number of straight flushes, students will generally easily recognize that there are 4 possible suits. The number of possible denomination-sets is less obvious to some. A particular issue with some students is that, for the denominations they will start with something like:

> *There are 13 possibilities for the first denomination since we haven't chosen one yet. Then the next denomination you select has to be within 4 of the first one.*

The problem with this line of reasoning? The order in which the five cards come is not relevant. While the definition of a straight flush involves the word *sequential*, when calculating probabilities we must still view the five cards as a *set*. For students who continue to struggle, the hint we often give at this point is:

> *What do you know if I tell you that you have a jack-high straight flush?*

The full process that we hope the students will eventually use to calculate:

- *Specify Suit.* There are 4 possible suits.
- *Specify Denominations by choosing the highest denomination.* Only the cards 5, 6, 7, 8, 9, T, J, Q, and K can be the highest denomination (only the ace may be used on either end, so for example QKA23 is not a straight flush; we have considered TJQKA to be a "royal flush" so it is not counted here). Thus there are 9 ways to pick the denomination set (since once we pick the highest card we must have the corresponding 4 underneath it).

Applying the multiplication rule to the above we now recognize that in a regular deck, there are $4 \times 9 = 36$ Straight Flushes.

Pinochle Deck (thought process and calculations)

Having already thought through the Royal Flush with this deck, and seeing the calculations for the regular deck for the straight flush, students should figure this one out with very minimal guidance:

- *Specify Suit.* There are 4 possible suits.

- *Specify Denominations by choosing the highest denomination.* There is only ONE possibility – the King – since the lowest card in this deck is 9.
- *Recognize at this point you have selected 10 cards; perhaps the KKQQJJTT99 of diamonds.* As with the Royal Flush, you must choose one of the two aces, one of the two kings, one of the two queens, etc. There are 2 choices available for each of the 5 denominations.

Applying the multiplication rule to the above we now recognize that in a pinochle deck, there are $4 \times 1 \times 2^5 = 128$ Straight Flushes.

B.3 Four-of-a-Kind

Definition: Four-of-a-kind is defined as 4 cards of one denomination and 1 card of a second denomination. As an example, the hand 9♣9♥9♦9♠Q♦ would be considered four-of-a-kind.

Regular Deck (thought process and calculations)

In counting the number of hands having the value four-of-a-kind, students should recognize that they must choose two denominations (one for the four cards that are matched and another for the card that is not matched). The key point is that students must be careful to recognize that the order of selection matters in this case! If we select denominations 9 and Q, there will be four 9s and one Q. This is different from selecting Q and 9, which would result in four queens and one 9. The full process:

- *Specify Denominations.* As discussed above, order matters. There are 13 denominations and we need two of them. Thus there are P(13,2) = 156 possibilities.
- *Specify Suits.* For the denomination of the matched cards we must take all four suits. For the unmatched card we must pick only one. It is a good idea for the students to view these as combinations: C(4,4) and C(4,1).

Again the key to this process is recognizing when order matters and when it does not. The denominations have *different functions* in this poker hand, so order matters when selecting them. But the cards in the hand are dealt all at once – so order will not matter when picking the suits. Applying the multiplication rule to the pieces above, there are $P(13,2) \times C(4,4) \times C(4,1) = 624$ different hands representing four-of-a-kind.

It is worth noting here that there are other thought processes by which one might arrive at the same answer. For example, 13 ways to pick the denomination for the four matched cards (suits are automatic) times 48 ways to pick the remaining card that doesn't match (since there are 48 cards left in the deck). This is completely valid, but may not be quite as helpful when students are trying to calculate for the pinochle deck.

Pinochle Deck (thought process and calculations)

Students should focus on the differences from the regular deck. They now need to specify two of six denominations; and then remember that there are eight cards of each denomination rather than four. If they used the suggested calculation above, the analogous calculation for the pinochle deck is $P(6,2)\times C(8,4)\times C(8,1)=16,800$. Some students will be surprised by the magnitude of this number, but since there are fewer ranks and more cards per rank, they should recognize that it is reasonable that four-of-a-kind is a much more common hand when using the pinochle deck.

If students attacked the regular-deck four-of-a-kind in a different way, instructors are encouraged to allow this, and to help the student see a similar approach using the pinochle deck. Corresponding to the example given above, we would have 6 ways to pick the denomination for the four matched cards. Selection from among the four suits is no longer enough – since there are two cards of each suit. The suits do not matter here (they cannot all be the same); hence the simplest way to count is to take the eight available cards and select four: there are C(8,4) ways to do this. Lastly, students must be careful to recognize that there are 40 (not 44) remaining cards that are eligible to fill the remaining card in the hand. Note that depending on the complexity of their original approach, it may be easier to show students how their original approach is the same as $P(13,2)\times C(4,4)\times C(4,1)=624$ so that they can then make a less awkward transition into the pinochle deck format.

B.4 Full House

Definition: A full house is defined as three cards of one denomination and two cards of a second (different) denomination. As an example, the hand A♣A♥A♦Q♠Q♦ would be classified as "aces full of queens".

Regular Deck (thought process and calculations)

When constructing the calculation for the number of full houses, it can be helpful for students to consider analogies to four-of-a-kind. Again they must choose two denominations (one for the set of three and another for the matched pair) and the order of selection is relevant (aces full of queens is a different hand from queens full of aces). Selection of the suits is also similar. The full process:

- *Specify Denominations.* As discussed above, order matters. There are 13 denominations and we need two of them (having different jobs). Thus there are P(13,2) = 156 possibilities.
- *Specify Suits.* The denomination selected for the set of three will need three of four suits chosen, while the pair will need two: C(4,3) and C(4,2).

Applying the multiplication rule, there are $P(13,2)\times C(4,3)\times C(4,2)=3,744$ different full houses.

Pinochle Deck (thought process and calculations)

The pinochle deck is perfectly analogous to the regular deck, and most students seem to find this one pretty easy. It is important to note that it is impossible for this hand to also be a flush (due to the set of three) but that will be a concern later on for this deck. The process:

- *Specify Denominations.* As discussed above, order matters. There are 6 denominations and we need two of them (having different jobs). Thus there are P(6,2) = 30 possibilities.
- *Specify Suits.* Again the pinochle deck has 8 cards of each denomination so we will have: C(8,3) and C(8,2).

Applying the multiplication rule, there are $P(6,2)\times C(8,3)\times C(8,2)=47,040$ different full houses.

An Intriguing Question

Obviously there are more full houses available in a pinochle deck than in a regular deck. Students are often surprised by this. Should this be obvious based on the makeup of the decks? This is a good question for students to consider (and perhaps write a short essay on the topic). Ultimately, it isn't 100% obvious. But the important points:

- Decreasing the number of denominations will decrease the number of full houses. P(13,2) is roughly five times larger than P(6,2).
- Increasing the number of cards in each rank will increase the number of full houses. C(8,3) is fourteen times larger than C(4,3).

By thinking about how changes to the deck affect the number of possibilities, it seems reasonable that students may gain a better understanding of how the numbers work. It is also valid to ask them how changing the deck affects the *probability of a full house* – for which they must also take into account how adding/removing cards from the deck affects the overall sample space.

B.5 Flush

Definition: A Flush is defined as five cards from the same suit (that are non-sequential – i.e. do not represent a Royal Flush or Straight Flush). The hands AK752♥ or KQT98♣ are both examples of flushes.

Regular Deck (thought process and calculations)

Counting the number of flushes is a relatively simple process for students using a regular deck. At this point, they should easily recognize that it is necessary to pick one of the four suits using C(4,1) and pick five denominations within that suit using C(13,5). Students may need to be reminded that the order of selection for these five cards is irrelevant (a hand is a hand), but generally they understand that at this point. Based on the computation of Royal and Straight Flushes, they also generally recognize that C(13,5) does nothing to prevent these – and hence the

40 Royal/Straight Flushes must be subtracted so that these hands are excluded from the count. The full calculation: $C(4,1)\times C(13,5)-40=5{,}108$.

Pinochle Deck (thought process and calculations)

This is where things start to get interesting. Students don't have much trouble extending the flush calculation to the pinochle deck. There are only 12 cards in each suit, so it is:

$$C(4,1)\times C(12,5)-256=2{,}912$$

For less advanced students, this is as far as they take it (and at this point, that is ok). More advanced students will notice that, in the pinochle deck, flushes may contain pairs (one or two). Some might even calculate the following hands:

- Two-pair flush: $\overbrace{C(4,1)}^{\text{SUIT}}\times\overbrace{C(6,2)\times C(4,1)}^{\text{DENOMINATIONS}}\times\overbrace{C(2,2)\times C(2,2)\times C(2,1)}^{\text{CARDS (2 of each suit)}}=480$
- One-pair flush: $\overbrace{C(4,1)}^{\text{SUIT}}\times\overbrace{C(6,1)\times C(5,3)}^{\text{DENOMINATIONS}}\times\overbrace{C(2,2)\times C(2,1)\times C(2,1)\times C(2,1)}^{\text{CARDS (2 of each suit)}}=1{,}920$
- No-pair flush: $\overbrace{C(4,1)}^{\text{SUIT}}\times\overbrace{C(4,3)}^{\text{DENOMINATIONS}}\times\overbrace{\left[C(2,1)\right]^5}^{\text{CARDS (2 of each suit)}}=512$

For the no-pair flush, the C(4,3) to choose denominations requires some explanation. One might as easily use C(6,5) – 2 for this term (all flushes minus straight flushes). However, the realization that a non-straight, high-card flush must include an ace and a nine in the pinochle deck leads to choosing three of the remaining four (KQJT) to go along with these two denominations.

It should be noted that generally we do not discuss the different "types" of flushes at this point unless a student brings them up. They will discover them later when they try to do two-pair, one-pair, and high card. In each case they must subtract the appropriate number to avoid overlap between the hand types.

B.6 Straight

Definition: A Straight is defined as five cards having sequential denominations (ace may be treated as high or low) that are not all of the same suit. As an example, T♥9♥8♣7♥6♣ would be a straight.

Regular Deck (thought process and calculations)

As students consider how to count the number of straights, they should recognize that the situation is most similar to the straight flush. In particular, the highest card in the straight must be 5, 6, 7, 8, 9, T, J, Q, K or A. And if we know the denomination of the highest card, then we know all denominations. The full process of calculation:

- *Specify Denominations by choosing the highest denomination.* There are 10 possibilities.

- *Specify Suits.* We must make sure that not all suits are the same – however this can be done by subtraction: there are $\left[\left[C(4,1)\right]^5 - C(4,1)\right]$ possibilities (the subtraction of C(4,1) excludes flushes).

Applying the multiplication rule to the above we now recognize that in a regular deck, there are $10\times\left[\left[C(4,1)\right]^5 - C(4,1)\right] = 10,200$ possible straights.

<u>Pinochle Deck</u> (thought process and calculations)

A similar process works for the pinochle deck – but the following changes affect the calculation:

- There are only two possibilities for the highest denomination.
- There are eight cards of each denomination including two in each suit, and because of this many more flushes to subtract.

The result: $2\times\left[\left[C(8,1)\right]^5 - C(4,1)\times\left[C(2,1)\right]^5\right] = 65,280$

B.7 Three-of-a-Kind

<u>Definition</u>: Three-of-a-kind is defined as 3 cards of one denomination and two other cards that do not match the original denomination nor each other. As an example, the hand 9♣9♥9♦J♠Q♦ would be considered three-of-a-kind (but 9♣9♥9♦Q♠Q♦ would be a full house).

<u>Regular Deck</u> (thought process and calculations)

In counting the number of hands having the value three-of-a-kind, students will likely start with their calculation from four-of-a-kind. A common mistake is to specify P(13,3) instead of P(13,2) to determine the denominations of the cards. Why doesn't this work? Again the key point here is that students must be careful to assess when order matters and when it does not! Consider:

- Three nines, a ten, and a jack is not the same hand as three tens, a nine, and a jack.
- It *is* the same hand as three nines, a jack, and a ten. Order matters when we select the set of three vs. the singles. But as we select the two singles order does not matter.

The full process:

- *Specify Denominations.* As discussed above, order matters in part. There are 13 choices for the set of three, and then conditionally C(12,2) choices for the denominations of the singles.
- *Specify Suits.* For the denomination of the set we must take three of four suits. For the unmatched cards we must pick only one. So we have $C(4,3)\times C(4,1)\times C(4,1)$.

Applying the multiplication rule to piece these together, there are 54,912 different hands of type "three-of-a-kind":

Note that another common hang-up for students is to select the set of three first and then choose the remaining two cards from the 49 remaining in the deck using C(49,2). It may be pointed out that this prevents neither the 4th card of the same denomination (to make 4 of a kind) nor a pair (to make a full house) among the remaining two cards. We want to prevent these things so as to avoid both circular reasoning (if the 4th card I select is the same denomination as the first three, then the first three didn't all need to be that denomination) and the need for exclusion (subtraction, as we often need to do with flushes).

Pinochle Deck (thought process and calculations)

Like the full house, since flushes are still not possible (students should be asked to think about the logic for this!) this is a very similar calculation using the pinochle deck. We need only remember that there are eight cards of each denomination rather than four (actual suits are not relevant since the flush isn't possible). The calculation:

$$6\times C(5,2)\times C(8,3)\times C(8,1)\times C(8,1)=215,040$$

There are 215,040 possible hands having exactly three of a kind.

B.8 Two Pair

Definition: Two-pair is defined as two cards of one denomination, two cards of a second denomination, and one card of a third denomination. For example, the hand A♣A♥K♦K♠Q♦ would be classified as "two pair: aces and kings".

Regular Deck (thought process and calculations)

The choice of denominations for two pair is actually congruent to the calculation for three-of-a-kind. If the pairs are to be, for example, 8's and 7's, it does not matter whether you select 8 or 7 first. But the remaining card being a jack is very different from one of the two pairs being jacks. So we have:

- *Specify Denominations.* As discussed above, order matters in part. There are C(13,2) choices for denominations of our pairs and then conditionally 11 choices for the denomination of the single. Observant students might note that we can decide the denomination of the single first: $C(13,2)\times 11=13\times C(12,2)$
- *Specify Suits.* For the pairs we must take two of four suits, while for the unmatched card we must pick only one. Note that in no case does the order of selection for suits make a difference to us. So we have $C(4,2)\times C(4,2)\times C(4,1)$.

Applying the multiplication rule, there are $C(13,2)\times 11\times C(4,2)\times C(4,2)\times C(4,1)=123,552$ different hands at the rank of two-pair.

<u>Pinochle Deck</u> (thought process and calculations)

The pinochle deck gets tricky when it comes to two-pair. The reason for this is a two-pair hand may also qualify as a flush (since there are two of every denomination in every suit). The most astute students will notice this, but *quite often we will allow students to go all the way through the high card and find that the results do not add up properly* before we give them hints in this direction.

The *first part* of the calculation is similar to the regular deck:

$$C(6,2)\times 4\times C(8,2)\times C(8,2)\times C(8,1)=376,320$$

The number of flushes containing two pair was calculated in section B.5. There are many more hands with two-pair than flushes, so based on this students should suggest that a hand containing both two pair and a flush be counted as a flush. Thus the 480 hands (see B.5 for details of calculation) that would qualify as both must be subtracted from the above total. There are 375,840 hands of rank "two-pair" in the pinochle deck.

Note that two-pair, one-pair, and high-card are often the most difficult for students to get – simply because they do not immediately think about the potential double-counting of the flushes. This is one aspect of the activity that requires more substantial critical thinking to account for the differences between the pinochle and regular decks (and also requires that students not proceed too hastily). We would encourage that students be allowed to figure this out on their own as much as possible – perhaps with only vague hints given as needed.

B.9 One Pair

<u>Definition</u>: One-pair is defined as two cards of one denomination, and three other cards of *different denominations* (from each other and also from the pair). For example, the hand A♣A♥K♦J♠9♦ would be classified as "a pair of aces".

<u>Regular Deck</u> (thought process and calculations)

Students can borrow from two-pair and three-of-a-kind on this one (and as such should almost certainly be asked to do this mostly on their own). Small hints might be given as needed. The overall process:

- *Specify Denominations.* Four different but order of selection matters only for pair vs. single. There are C(13,1) ways to choose the denomination for the pair, and conditionally C(12,3) ways to choose the denominations for the singles. As is sometimes the case with three of a kind, students need to remember that C(49,3) isn't acceptable to choose the three odd cards, as the hand might then wind up having more than just a pair.
- *Specify Suits.* For the pair we must take two of four suits, while for the unmatched cards we must pick only one. Note that in no case does the order of selection for suits make a difference to us. So we will have $C(4,2)\times\left[C(4,1)\right]^3$.

Applying the multiplication rule, there are $C(13,1)\times C(12,3)\times C(4,2)\times\left[C(4,1)\right]^3=1,098,240$ different hands at the rank of one-pair.

Pinochle Deck (thought process and calculations)

As was the case with two-pair, the pinochle deck is a bit tricky for one-pair since some hands containing one pair will also qualify as a flush. The *first part* of the calculation is of course nearly identical to the regular deck:

$$C(6,1)\times C(5,3)\times C(8,2)\times\left[C(8,1)\right]^3=860,160$$

The number of flushes containing one pair was calculated in section B.5. Again there are many more hands with one-pair than flushes; based on this fact students should suggest that a hand containing both one-pair and a flush be counted as a flush. Thus the 1,920 hands (see B.5 for details of calculation) that would qualify as both must be subtracted from the above total. There are 858,240 hands of rank "one-pair" in the pinochle deck.

B.10 High-Card

Definition: A High-Card hand is defined to be any hand that does not fall into any of the other categories. In other words, it contains five cards of different denominations such that no straights or flushes are formed. As an example, the hand A♣Q♥T♦6♠3♦ would be classified as an "ace-high".

Regular Deck (thought process and calculations)

First a note on calculation: students often want to apply the complement rule, subtracting everything else from C(52,5), to calculate the number of ways to get this hand. While this is a valid method, it is one that we choose not to allow. Instead, students are encouraged to total all hand-types at the end to make sure they get C(52,5). If they do not, they are encouraged to go back and find their mistake.

Calculating the number of high-card hands directly might be the most difficult of all calculations for the students to see. They must recognize that leaving out straights and flushes still involves some subtraction. There are multiple methods of getting there – here is a favorite of the authors:

- *Specify Denominations.* Five different and not in a sequence. To obtain this we can take any five of the 13 and then subtract the number of sequences (10). So there are C(13,5) – 10 ways to select the denominations.
- *Specify Suits.* At least two different suits must be used. There are $\left[C(4,1)\right]^5$ ways to arbitrarily select the suits, but of course four of those ways result in a flush. Thus there are $\left[C(4,1)\right]^5-4$ ways to select the suits.

Applying the multiplication rule, there are $\left[C\left(13,5\right)-10\right]\left[\left[C\left(4,1\right)\right]^{5}-4\right]=1,302,540$ different hands at the rank of high-card.

Pinochle Deck (thought process and calculations)

One of the main reasons we like the above method is because it is easy to generalize the method to the pinochle deck, where there are only

$$\left[C\left(6,5\right)-2\right]\left[\left[C\left(8,1\right)\right]^{5}-4*\left[C\left(2,1\right)\right]^{5}\right]=130,560$$

possible hands having only a "high-card". The only major difference is in the $\left[C\left(2,1\right)\right]^{5}$ which accounts for the fact that there are two cards of every denomination in every suit.

More Intriguing Questions

There are only about 1/10 the number of high-card hands in the pinochle deck! Should this be surprising to us? Again this type of questioning allows students to consider the manner in which changes to the deck affect the number of ways to get a hand. Since we have fewer denominations (making five different less likely) and more cards of each denomination (making matches more likely), this shouldn't surprise students too much. They might then consider whether or not this should still be the "worst-ranked" hand.

B.11 Five-of-a-Kind

Definition: Five-of-a-kind is defined as 5 cards of one denomination. As an example, the hand 9♣9♥9♦9♠9♦ would be considered five-of-a-kind. Naturally this is not possible with a regular deck (since there are only four cards of each denomination).

Pinochle Deck (thought process and calculations)

The calculation is fairly simple (in fact many students will do it before doing four-of-a-kind). One must choose denomination and five cards from that denomination (no flushes are possible). Hence there are $C\left(6,1\right)\times C(8,5)=336$ possible hands in the category five-of-a-kind.

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Jacqueline Wroughton
Northern Kentucky University
Nunn Drive
MP 422
Highland Heights, KY 41099
wroughtonj1@nku.edu
Phone: (859)572-5331

Joseph Nolan
Northern Kentucky University
Nunn Drive
MP 418
Highland Heights, KY 41099
nolanj1@nku.edu
